MATH 245 F23, Exam 3 Solutions

1. Carefully define the following terms: intersection, antisymmetric

   For any sets R, S, their intersection is the set $\{x : x \in R \wedge x \in S\}$. Given a relation R on set S, we say R is antisymmetric if $\forall x, y \in S, (x = y) \vee (x \not R y) \vee (y \not R x)$. ALTERNATE: $\forall x, y \in S, (xRy \wedge yRx) \to x = y$.

2. Carefully state the following theorems: Commutativity Theorem (for sets), De Morgan's Law Theorem (for sets)

   The Commutativity Theorem states that, for any sets R, S, we must have (a) $R \cup S = S \cup R$; and (b) $R \cap S = S \cap R$; and (c) $R \Delta S = S \Delta R$. The De Morgan's Law Theorem states that, for any sets R, S, U with $R \subseteq U$ and $S \subseteq U$, we have both (a) $(R \cup S)^c = R^c \cap S^c$; and (b) $(R \cap S)^c = R^c \cup S^c$.

3. Let $R = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 2y\}, S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 3y\}, T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 6y\}$. Prove or disprove that $R \cap S = T$.

   The statement is true, and must be proved in two parts.
   $\boxed{T \subseteq R \cap S}$ (the easier part): Let $x \in T$. Then there is some $y \in \mathbb{Z}$ with $x = 6y$. We have $x = 2(3y)$, and $3y \in \mathbb{Z}$, so $x \in R$. We also have $x = 3(2y)$, and $2y \in \mathbb{Z}$, so $x \in S$. By conjunction, $x \in R \wedge x \in S$, so $x \in R \cap S$.
   $\boxed{R \cap S \subseteq T}$ (the harder part): Let $x \in R \cap S$. Then $x \in R \wedge x \in S$, so by simplification twice, $x \in R$ and $x \in S$. Hence there are $y, y' \in \mathbb{Z}$ with $x = 2y$ and $x = 3y'$. Now $2y = 3y'$, so $2|3y'$. Since 2 is prime, either $2|3$ (it doesn't) or $2|y'$. Hence there is some $z \in \mathbb{Z}$ with $2z = y'$. So, $x = 3y' = 3(2z) = 6z$, where $z \in \mathbb{Z}$. This proves $x \in T$.

4. Let R, S be sets. Prove that $R \Delta S \subseteq R \cup S$.

   Let $x \in R \Delta S$. Then $(x \in R \wedge x \notin S) \vee (x \notin R \wedge x \in S)$. We have two cases:
   $\boxed{x \in R \wedge x \notin S}$: By simplification, $x \in R$. By addition, $x \in R \vee x \in S$, so $x \in R \cup S$.
   $\boxed{x \notin R \wedge x \in S}$: By simplification, $x \in S$. By addition, $x \in R \vee x \in S$, so $x \in R \cup S$.
   In both cases, $x \in R \cup S$.

   NOTE: If you used $(x \in R \wedge x \notin S) \vee (x \in S \wedge x \notin R)$, that's not the definition, but it's close enough that I don't care. If you used $R \Delta S = (R \cup S) \setminus (R \cap S)$ or $R \Delta S = (R \setminus S) \cup (S \setminus R)$, that's a theorem and must be cited.

5. Let S, T, U be sets with $S \subseteq U$ and $T \subseteq U$. Prove that $S \cap T^c \subseteq S \setminus T$.

   Let $x \in S \cap T^c$. Then $x \in S \wedge x \in T^c$, so by simplification twice $x \in S$ and $x \in T^c$. Now $x \in U \setminus T$, so $x \in U \wedge x \notin T$. By simplification, $x \notin T$. We now use conjunction to get $x \in S \wedge x \notin T$. Hence, $x \in S \setminus T$.

6. Let A, B be sets with $7 \in A$ and $A \times A = A \times B$. Prove that $A = B$.

   $\boxed{A \subseteq B}$: Let $x \in A$ be arbitrary. Then $(7, x) \in A \times A$. Since $A \times A = A \times B$, we have $(7, x) \in A \times B$, so $x \in B$.
   $\boxed{B \subseteq A}$: Let $x \in B$ be arbitrary. Then $(7, x) \in A \times B$. Since $A \times A = A \times B$, we have $(7, x) \in A \times A$, so $x \in A$.

   NOTE: It is essential that $7 \in A$, or at least that A is nonempty. Otherwise, if $A = \emptyset$, then $A \times A = A \times B = \emptyset$, for any set B. (and the statement is false, since $A \neq B$)
   If your proof didn't use this fact, you have a serious problem!

7. Find some nonempty set S that satisfies $S \subseteq 2^S$. Give both S and 2^S in list notation, and justify the relationship.

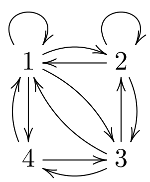
Note that all the elements of 2^S are sets, so any solution must have all elements of S be sets as well (since those elements are among the elements of 2^S). There are two basic approaches:

SOLUTION 1: $S = \{\emptyset\}, 2^S = \{\emptyset, S\}$. Note that S contains only one element, $\emptyset$, and this is an element of 2^S as well. This solution uses the fact that 2^S always contains the empty set.

SOLUTION 2: $S = \{S\}, 2^S = \{\emptyset, S\}$. Note that S contains only one element, namely S, which is also contained in 2^S. This solution uses the fact that 2^S always contains S.

This second solution uses a crazy set that is an element of itself! This sort of tomfoolery is forbidden by the "Axiom of Regularity", which one would have to set aside for this solution. Needless to say solution 2 is more controversial than solution 1, but I cheerfully accept either.

8. Let $S = \{1, 2, 3, 4\}$ and consider the relation $R = \{(x, y) : xy|12\}$, on S. Draw this relation as a digraph, and determine (with justification) whether or not it is (a) reflexive; (b) irreflexive; (c) trichotomous; (d) transitive.



The digraph should have every possible edge, except for the four edges $(3, 3), (4, 4), (2, 4), (4, 2)$.
Reflexive: No, because $(3, 3) \notin R$. Irreflexive: No, because $(2, 2) \in R$.
Trichotomous: No, because $(2, 4) \notin R$ and $(4, 2) \notin R$.
Transitive: No, because $(2, 3) \in R$ and $(3, 4) \in R$, but $(2, 4) \notin R$.

Some students confused $xy|12$ with $xy = 12$, which completely ruined the digraph, and also (unfortunately for them) trivialized problem 9.

9. Let $S = \{1, 2, 3, 4\}$ and consider the relation $R = \{(x, y) : xy|12\}$, on S. Prove or disprove that $R^{(2)} = S \times S$.

The statement is true. First, $R^{(2)} \subseteq T \times T$ is true since $R^{(2)} = R \circ R$ is a relation on S. For the other direction, let $(x, y) \in S \times S$ be arbitrary. Note that each element of S divides 12. In particular $x \cdot 1 = x|12$, so $(x, 1) \in R$. Also, $1 \cdot y = y|12$, so $(1, y) \in R$. Since $(x, 1), (1, y) \in R$, we have $(x, y) \in R^{(2)}$.

10. Prove or disprove that $|\mathbb{Z}| \leq |\mathbb{N} \times \mathbb{N}|$.

The statement is true, and a proof requires a pairing between $\mathbb{Z}$ and some subset of $\mathbb{N} \times \mathbb{N}$. Many pairings are possible; here are three examples. Remember that $0 \notin \mathbb{N}$.

SOLUTION 1: For all $n \in \mathbb{Z}$, we have the pairing $n \leftrightarrow \begin{cases} (1, n) & n > 0 \\ (2, -n) & n < 0 \\ (3, 1) & n = 0 \end{cases}$.

SOLUTION 2: For all $n \in \mathbb{Z}$, we have the pairing $n \leftrightarrow \begin{cases} (1, n) & n > 0 \\ (2, 1-n) & n \leq 0 \end{cases}$.

SOLUTION 3: For all $n \in \mathbb{Z}$, we have the pairing $n \leftrightarrow \begin{cases} (1, 2n) & n > 0 \\ (1, 1-2n) & n \leq 0 \end{cases}$.

When grading this, I am looking for a pairing that works, and correct notation throughout. I do not require a *proof* that the pairing works, i.e. is a bijection. You also don't need to specify exactly which subset of $\mathbb{N} \times \mathbb{N}$ is paired off with $\mathbb{Z}$.

SOLUTION 4: A student solution suggested the following clever approach. By Theorem 9.16, $|\mathbb{Z}| = |\mathbb{N}|$. Hence, there is a pairing $z \leftrightarrow n$, for every $z \in \mathbb{Z}$ and every $n \in \mathbb{N}$. Now, we chain this together with the pairing $n \leftrightarrow (n, n)$, which pairs $\mathbb{N}$ with a subset of $\mathbb{N} \times \mathbb{N}$. Hence, together, this gives $z \leftrightarrow n \leftrightarrow (n, n)$, a pairing between $\mathbb{Z}$ and a subset of $\mathbb{N} \times \mathbb{N}$ (namely, the diagonal subset!)